Dialogue Social Science Review (DSSR)
www.thedssr.com

ISSN Online: 3007-3154
ISSN Print: 3007-3146

DIALOGUE SOCIAL SCIENCE REVIEW

Vol. 3 No. 8 (August) (2025)

Homomorphisms between Pythagorean Multi Anti-Fuzzy Rings

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Abstract
One of the important extensions of both the fuzzy set as well as the intuitionistic fuzzy set is the Pythagorean fuzzy sets which provides a flexible framework to deal with uncertainty and ambiguity. Rings are extensively studied within the modern algebra which helps in dealing with some complex problems of modern science and technology. In this article we propose the homomorphism between Pythagorean multi anti fuzzy rings. We also discuss some of its characteristics and use this construction to further study the image, pre image and some other properties of the homomorphism defined within this framework.
Keywords: Homomorphisms, Pythagorean Multi Anti-Fuzzy Rings

1. Introduction

Since its introduction by Lotfi A. Zadeh in 1965 (Zadeh, 1965), fuzzy set theory has become a groundbreaking idea in mathematics and has found useful applications in a number of domains, such as pattern recognition, artificial intelligence, decision-making, and control systems. Fuzzy set theory's primary attraction is its capacity to deal with ambiguity and uncertainty, which traditional set theory finds difficult to manage. The idea of intuitionistic fuzzy sets (ĬƑSs), first proposed by Atanassov in 1983 (Atanassov, 1986), expanded fuzzy set theory by adding two new parameters: the degree of membership and the degree of non-membership. By adding multi-dimensional membership functions. The ĬƑS were further generalized by Yager by the construction of the Pythagorean Fuzzy Sets (PFSs) (Yager, 2013). Sabu Sebastian developed multi-fuzzy set theory (Sabu, 2011), which expanded on these fundamental concepts and further generalized fuzzy sets. This development made it possible to model systems with several criteria or characteristics, which makes it especially applicable in fields like multi-criteria optimization and multi-objective decision-making. R. Muthuraj and S.

Dialogue Social Science Review (DSSR)
www.thedssr.com

ISSN Online: 3007-3154
ISSN Print: 3007-3146

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By proposing fuzzy subgroups, Rosenfeld (Rosenfeld, 1971) expanded fuzzy set theory to algebraic structures. WJ Liu proposed the ring structure and its ideals under the fuzzy framework (Liu, 1982) which is further studied by well-known researchers in the contexts of multi fuzzy (Al-Husban, 2022), intuitionistic fuzzy (Marashdeh, 2001) and Pythagorean fuzzy settings (Razaq, 2023). Intuitionistic multi fuzzy near-rings and ideals associated with it were studied by Batool et. al. (Batool, 2023). Balamurugan (Muthuraj, 2013) developed the idea of multi-anti fuzzy subgroups, which integrate anti-membership functions into multi-fuzzy set theory, as an extension of these ideas. Recently the characterization of intuitionistic multi anti-fuzzy was done by Muthuraj (Muthuraj, 2022) in which the Cartesian product and homomorphisms were focused.
The work specifically focusses on defining and analyzing the basic features of the the homomorphism of between Pythagorean multi-anti fuzzy(PMĂƑ) rings. Moreover, we investigate the characteristics of this novel construction by defining kernel and image under homomorphism and anti-homomorphism, emphasizing the importance of these aspects in comprehending the structural connections among these fuzzy systems.

2. Preliminaries

This section is devoted to describe some of the pre-requisite notions which are used in this study and without having an acquaintance of these concepts it will be difficult to understand the text.

Definition 2.0.1. Suppose a non-empty set E then the fuzzy subset (FSS) M of E can be defined as

$$M = \{(\alpha,\ \mho(\alpha)) : \alpha \epsilon E \text{ and } \mho(\alpha) : E \to [0,1].$$

Example 2.0.2.

We can define a FSS N for the set $X = \{1,\ 2,\ 3,\ 4,\ 5\}$ as

$$N = \{(1,\ .5), (2,\ .2), (3,\ .4), (4,\ .3), (5,\ .1)\}$$

If $X = \{15,\ 25,\ 35,\ 45,\ 55\}$ shows ages then the FSSs "$young$" and " $Adult$" will b

$$Young = \{(15,\ .9)(25,\ .8)(35,\ .5)(45,\ 1)(55,\ 0)\}$$
$$Adult = \{(15,\ 0)(25,\ .5)(35,\ .8)(45,\ 1)(55,\ 1)\}$$

Definition 2.0.3. Suppose A is a set then the ĬƑS described on A can be represented as $M = \{(\alpha,\ \mho(\alpha),\ \Omega(\alpha)) : \alpha \epsilon M,$ $\mho(\alpha) : A \to [0,\ 1]$ and $\Omega(\alpha) : \to [0,\ 1]\}$ such that $0 \le \mho(\alpha) + \Omega(\alpha) \le 1$.

Definition 2.0.4. The Pythagorean fuzzy set M constructed on the discourse set A can be represented as $M = \{(\alpha,\ \mho(\alpha),\ \Omega(\alpha)) : \alpha \epsilon M, \mho(\alpha) : A \to [0,\ 1]$ and $\Omega(\alpha) : \to [0,\ 1]\}$
Such that $0 \le {\mho_1}^2(\alpha) + {\Omega_1}^2(\alpha) \le 1$

Example 2.0.5 For the ground set $A = \{1,\ 2,\ 3\}$ the set below describes the PFSs

$$M = \{(1,\ 0.2,\ 0.9), (2,\ 0.3,\ 0.4), (3,\ 0.8,\ 0.2)\}$$

Definition 2.0.6. Suppose a non- empty set X. A multi-fuzzy set M is defined as

$$M = \{(\alpha,\ \mho_1(\alpha),\ \mho_2(\alpha),\ \mho_3(\alpha),\ \mho_4(\alpha), \dots\dots) : \alpha \epsilon X\},$$

where, $\mho_\mathfrak{i}(\alpha) \to [0,\ 1]$ for all $\mathfrak{i}$. For instance, set written below is multi fuzzy

$$M = \{(a, \{.1,\ .2,\ .4\}, (b, \{.3,\ .4,\ .001\}), (c,\ \{.2,\ .1,\ .25\})\}$$

Definition 2.0.7. An Intuitionistic multi-fuzzy set M is defined as

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www.thedssr.com

ISSN Online: 3007-3154
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DIALOGUE SOCIAL SCIENCE REVIEW

Vol. 3 No. 8 (August) (2025)

$$M = \{(\alpha, \mho_1(\alpha), \mho_2(\alpha), \mho_3(\alpha), \ldots, \mho_k; \Omega_1(\alpha), \Omega_2(\alpha), \Omega_3(\alpha), \ldots, \Omega_k(\alpha)) : \alpha \in M\}$$

where, $\mho_k(\alpha) \to [0, 1]$ for all k and $\Omega_k(\alpha) \to [0, 1]$.

Definition 2.0.8. Take X to be the set of discourse then one can describe the Pythagorean multi fuzzy set

$$P_M^F = \{(\alpha, \mathcal{C}\Omega(\alpha), \mathcal{C}\mho(\alpha)) / \alpha \in X\}$$

is described by two maps $\mathcal{C}\Omega(\alpha): X \to Q$ and $\mathcal{C}\mho(\alpha): X \to Q$, labeling as functions of count membership and count non-membership, where the set Q exhibits family of all possible crisp multi-sets produced from the interval [0,1] and for each $\alpha \in X$, $\mathcal{C}\Omega(\alpha)$ will be a decreasingly ordered progression such as $(\mho_1(\alpha))^2 \geq (\mho_2(\alpha))^2 \geq (\mho_3(\alpha))^2 \geq \ldots \geq (\mho_k(\alpha))^2$ and $\mathcal{C}\mho(\alpha)$ could be of any order $\left(\Omega_1(\alpha)\right)^2, \left(\Omega_2(\alpha)\right)^2, \left(\Omega_3(\alpha)\right)^2, \ldots, \left(\Omega_k(\alpha)\right)^2$. For each $\alpha \in X, 0 \leq \left(\mathcal{C}\Omega(\alpha)\right)^2 + \left(\mathcal{C}\mho(\alpha)\right)^2 \leq 1$.

Example 2.0.9. Let $M = \{\sigma, \rho, \eta\}$ then $\mathcal{A}$ is intuitionistic fuzzy multi-set over $\mathcal{N}$ with count functions:

$$\mathcal{C}\Omega(\alpha) = \begin{cases} 0.3, 0.3, 0.5 & if\ \alpha = \sigma \\ 0.02, 0.02, 0.02 & if\ \alpha = \rho \\ 1, 1, 0.6 & if\ \alpha = \eta \end{cases}$$

$$\mathcal{C}\mho(\alpha) = \begin{cases} 0.5,0.5,0.3 & if\ \alpha = \sigma \\ 0.8,0.8,0.8 & if\ \alpha = \rho \\ 0.4 & if\ \alpha = \eta \end{cases}$$

Definition 2.0.10. Given two Pythagorean fuzzy multi-sets A and B over X with maps $\mathcal{C}\Omega_A(\alpha)$ and $\mathcal{C}\Omega_B(\alpha)$ labeling as functions of count membership of A and B and $\mathcal{C}\mho_A(\alpha)$ and $\mathcal{C}\mho_B(\alpha)$ showcases count non-membership, then:

i. $A \subseteq B$ if $\left(\mathcal{C}\Omega_A(\alpha)\right)^2 \leq \left(\mathcal{C}\Omega_B(\alpha)\right)^2$ and $\left(\mathcal{C}\mho_A(\alpha)\right)^2 \leq \left(\mathcal{C}\mho_B(\alpha)\right)^2\ \forall\ \alpha \in X$

ii. $A = B$ if $\left(\mathcal{C}\Omega_A(\alpha)\right)^2 = \left(\mathcal{C}\Omega_B(\alpha)\right)^2$ and $\left(\mathcal{C}\mho_A(\alpha)\right)^2 = \left(\mathcal{C}\mho_B(\alpha)\right)^2\ \forall\ \alpha \in X$.

iii. $\mathcal{C}\Omega_{A\cap B}(\alpha) = \min\left\{\left(\mathcal{C}\Omega_A(\alpha)\right)^2, \left(\mathcal{C}\Omega_B(\alpha)\right)^2\right\}$ and $\mathcal{C}\mho_{A\cap B}(\alpha) = \max\{\left(\mathcal{C}\mho_A(\alpha)\right)^2, \left(\mathcal{C}\mho_B(\alpha)\right)^2\}$.

iv. $\mathcal{C}\Omega_{A\cup B}(\alpha) = \max\left\{\left(\mathcal{C}\Omega_A(\alpha)\right)^2, \left(\mathcal{C}\Omega_B(\alpha)\right)^2\right\}$and $\mathcal{C}\mho_{A\cup B}(\alpha) = \min\left\{\left(\mathcal{C}\mho_A(\alpha)\right)^2, \left(\mathcal{C}\mho_B(\alpha)\right)^2\right\}$.

v. Complement of an intuitionistic fuzzy multi-set is defined as;

$$A^c = \{< \alpha, \left(\mathcal{C}\mho_A(\alpha)\right)^2, \left(\mathcal{C}\Omega_A(\alpha)\right)^2 > / \alpha \in X\}$$

Definition 2.0.11. Let M be a fuzzy set on a ring Ř. Then M is a fuzzy subring of Ꞡ if and only if

i. $\mho_M(\alpha - ß) \geq \min\{\mho_M(\alpha), \Omega_M(ß)\}$

ii. $\mho_M(\alpha . ß) \geq \min\{\mho_M(\alpha), \Omega_M(ß)\}$

Definition 2.0.12. Suppose a ring Ř, and $M = \left\{\left(\alpha, \Omega_M(\alpha), \mho_N(\alpha)\right) | \alpha \in Ř\right\}$ be Pythagorean fuzzy ring over Ř, if the given conditions are holds,

i. $\left(\Omega_M(\alpha - ß)\right)^2 \geq min\left\{\left(\Omega_M(\alpha)\right)^2, \left(\Omega_M(ß)\right)^2\right\}$

ii. $\left(\Omega_M(\alpha ß)\right)^2 \geq min\left\{\left(\Omega_M(\alpha)\right)^2, \left(\Omega_M(ß)\right)^2\right\}$

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DIALOGUE SOCIAL SCIENCE REVIEW

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iii. $\left(Ʊ_N(\alpha - ß)\right)^2 \leq \max\left\{\left(Ʊ_N(\alpha)\right)^2 \;, \left(\left(Ʊ_N(ß)\right)^2\right\}$

iv. $\left(Ʊ_N(\alpha ß)\right)^2 \leq max\left\{\left(Ʊ_N(\alpha)\right)^2 \;, \left(\left(Ʊ_N(ß)\right)^2\right\}, \quad \forall\ \alpha, ß\, \epsilon\, Ř$

Definition 2.0.13. Suppose a ring Ř, and $M = \{(\alpha, \Omega_M(\alpha),\ Ʊ_N(\alpha))|\alpha \in Ř\}$ be Pythagorean fuzzy ring over Ř, if the given conditions are holds,

i. $\left(\Omega_M(\alpha - ß)\right)^2 \leq max\,\{\left(\Omega_M(\alpha)\right)^2, \left(\Omega_M(ß)\right)^2\}$

ii. $\left(\Omega_M(\alpha ß)\right)^2 \leq max\,\left\{\left(\Omega_M(\alpha)\right)^2, \left(\Omega_M(ß)\right)^2\right\}$

iii. $\left(Ʊ_N(\alpha - ß)\right)^2 \geq \min\left\{\left(Ʊ_N(\alpha)\right)^2 \;, \left(\left(Ʊ_N(ß)\right)^2\right\}$

iv. $\left(Ʊ_N(\alpha ß)\right)^2 \geq min\left\{\left(Ʊ_N(\alpha)\right)^2 \;, \left(\left(Ʊ_N(ß)\right)^2\right\}, \quad \forall\ \alpha, ß\, \epsilon\, Ř$

3. Main Results

Here we discussed about the properties of homomorphic and anti-homomorphic, anti-image and ant-pre-image of PMĄ̌Ƒ subring of a ring Ř.

Definition 3.1.1

Suppose $Ř_1$and $Ř_2$ be rings. Suppose $Ǥ = \{(\alpha, Ʊ_M(\alpha), Ʊ_N(\alpha))/\alpha\, \epsilon\, Ř_1\}$ and $Ȟ = \{(ß, Ʊ_O(ß), Ʊ_P(ß))/ß\, \epsilon\, Ř_2\}$ are ĬMƑ subsets on a ring $Ř_1$and $Ř_2$ correspondingly. Suppose a mapping $f: Ř_1 \to Ř_2$ then $f(Ǥ) = \{(ß, f(Ʊ_M(ß)), f(Ʊ_N(ß)))/ß\, \epsilon\, Ř_2\}$ denoted as anti-image of Ǥ, then $f(Ǥ)$ is an ĬMƑ subset of $Ř_2, \forall\, ß\, \epsilon\, Ř_2$.

$$f(Ʊ_M(ß)) = \begin{cases} inf\{Ʊ_M^2(\alpha): \alpha\, \epsilon\ \ f^{-1}(ß)\}, & if\ \ f^{-1}\,(ß) \neq \emptyset \\ 1, & otherwise \end{cases}$$

$$f(Ʊ_N(ß)) = \begin{cases} sup\{Ʊ^2{}_N(\alpha): \alpha\, \epsilon\ \ f^{-1}(ß)\}, & if\ \alpha\, \epsilon\, f^{-1}(ß) \neq \emptyset \\ 0, & otherwise \end{cases}$$

$f^{-1}(Ȟ) = \{(\ f^{-1}(\alpha),\ f^{-1}\, Ʊ_O(\alpha),\ f^{-1}Ʊ^2{}_P(\alpha))/\ \alpha\, \epsilon\, Ř1\}\ is$ an anti-pre-image of Ȟ under f and also an ĬMƑ subset of $Ř_1, \forall\, \alpha\, \epsilon\, Ř_1, f^{-1}(Ʊ^2{}_O(\alpha)) = Ʊ^2{}_O(f(\alpha))$ and $f^{-1}(Ʊ^2{}_P(\alpha)) = Ʊ^2{}_P(f(\alpha))$.

Theorem 3.1.2:

Suppose $Ř_1$and $Ř_2$ be two rings. Suppose $f: Ř_1 \to Ř_2$ is a homomorphism onto rings. Suppose $Ǥ = \{(\alpha,\ Ʊ_M(\alpha),\ Ʊ_N(\alpha))/\alpha\, \epsilon\, Ř_1\}$ be an ĬMĄ̌Ƒ subring of $Ř_1$then $f(Ǥ)$ is an ĬMĄ̌Ƒ subring of $Ř_2$, if Ǥ has an inf property and Ǥ is f-invariant.

Proof: Suppose $Ǥ = \{(\alpha,\ Ʊ_M(\alpha),\ Ʊ_N(\alpha))/\alpha\, \epsilon\, Ř_1\}$ be an ĬMĄ̌Ƒ subring of $Ř_1$. Then, $f(Ǥ) = \{\, ß,\ f(\,Ʊ^2{}_M(ß)),\ f(Ʊ^2{}_N(ß)))/ß\, \epsilon\, Ř_2\}$.

There exist $\alpha, ß\, \epsilon\, Ř_1$such that $f(\alpha), f(ß)\, \epsilon\, Ř_2$,

i.

$$\begin{aligned} &= Ʊ^2{}_M(\alpha -\ ß) \\ &\leq \max\{\,Ʊ^2{}_M(\alpha), Ʊ^2{}_M(ß)\} \\ &= \max\{(f(Ʊ^2{}_M))(f(\alpha)), (f(Ʊ^2{}_M))(f(ß))\} \end{aligned}$$

$$(f(Ʊ^2{}_M))(f(\alpha) -\ f(ß) \leq \max\{(f(Ʊ^2{}_M))(f(\alpha)), (f(Ʊ^2{}_M))(f(ß))\}.$$

ii.

$$\begin{aligned} &= Ʊ^2{}_M(\alpha ß) \\ &\leq \max\{\,Ʊ^2{}_M(\alpha), Ʊ^2{}_M(ß)\} \\ &= \max\{(f(Ʊ^2{}_M))(f(\alpha)), (f(Ʊ^2{}_M))(f(ß))\} \end{aligned}$$

ISSN Online: 3007-3154
ISSN Print: 3007-3146

DIALOGUE SOCIAL SCIENCE REVIEW

Vol. 3 No. 8 (August) (2025)

$$(f(Ʊ^2{}_M))f(\alpha)f(ß)) \leq \max\{(f(Ʊ^2{}_M))(f(\alpha)), (f(Ʊ^2{}_M))(f(ß))\}.$$

iii. $(f(Ʊ^2{}_N))(f(\alpha) - f(ß)) = (f(Ʊ^2{}_N))(f(\alpha - ß)),$

$$= Ʊ^2{}_N(\alpha - ß)$$
$$\geq \min\{Ʊ^2{}_N(\alpha), Ʊ^2{}_N(ß)\}$$
$$= \min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}$$
$$(f(Ʊ^2{}_N))(f(\alpha) - f(ß) \geq \min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}.$$

iv. $(f(Ʊ^2{}_N))(f(\alpha)\, f(ß)) = (f(Ʊ^2{}_N))(f(\alpha ß)),$

$$= Ʊ^2{}_N(\alpha ß)$$
$$\geq \min\{Ʊ^2{}_N(\alpha), Ʊ^2{}_N(ß)\}$$
$$= \min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}$$
$$(f(Ʊ^2{}_N))f(\alpha)f(ß)) \geq \min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}.$$

Hence, f(Ǥ) is an PMĂƑ subring of $Ř_2$.

Theorem 3.1.3.

Suppose $Ř_1$ and $Ř_2$ be two rings. Suppose $f: Ř_1 \to Ř_2$ is a homomorphism onto rings. Suppose $Ȟ = \{(ß, Ʊ_O(ß), Ʊ_P(ß))/ß \in Ř_2\}$ be an PMĂƑ subring of $Ř_2$ then $f^{-1}(Ȟ)$ is an PMĂƑ subring of $Ř_1$.

Proof

$Ȟ = \{(ß, Ʊ_O(ß), Ʊ_P(ß))/ß \in Ř_2\}$ be an ĬMĂƑ subring of $Ř_2$.

Then, $f^{-1}(Ȟ) = \{(\alpha, f^{-1}\, Ʊ^2{}_O(\alpha), f^{-1}\, Ʊ^2{}_P(\alpha))/ \alpha \in Ř_1\}$.

For any $\alpha, ß \in Ř_1, f(\alpha), f(ß) \in Ř_2,$

(i) $f^{-1}(Ʊ^2{}_O))(\alpha - ß)) = Ʊ^2{}_O(f(\alpha - ß))$

$$= Ʊ^2{}_O(f(\alpha), f(ß))$$
$$\leq \min\{(Ʊ^2{}_O(f(\alpha)), Ʊ^2{}_O(f(ß))\}$$
$$= \min\{(f^{-1}(Ʊ^2{}_O))(\alpha), (f^{-1}(Ʊ^2{}_O))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_O))(\alpha - ß) \leq \min\{(f^{-1}(Ʊ^2{}_O))(\alpha), (f^{-1}(Ʊ^2{}_O))(ß)\}.$$

(ii) $(f^{-1}(Ʊ^2{}_O))(\alpha ß)) = Ʊ^2{}_O(f(\alpha ß))$

$$= Ʊ^2{}_O(f(\alpha), f(ß))$$
$$\leq \min\{(Ʊ^2{}_O(f(\alpha)), Ʊ^2{}_O(f(ß))\}$$
$$= \min\{(f^{-1}(Ʊ^2{}_O))(\alpha), (f^{-1}(Ʊ^2{}_O))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_O))(\alpha ß) \leq \min\{(f^{-1}(Ʊ^2{}_O))(\alpha), (f^{-1}(Ʊ^2{}_O))(ß)\}.$$

(iii) $f^{-1}(Ʊ^2{}_P))(\alpha - ß)) = Ʊ^2{}_P(f(\alpha - ß))$

$$= Ʊ^2{}_P(f(\alpha - ß)$$
$$= Ʊ^2{}_P(f(\alpha), f(ß))$$
$$\leq \max\{(Ʊ^2{}_P(f(\alpha)), Ʊ^2{}_P(f(ß))\}$$
$$= \max\{(f^{-1}(Ʊ^2{}_P))(\alpha), (f^{-1}(Ʊ^2{}_P))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_P))(\alpha - ß) \leq \max\{(f^{-1}(Ʊ^2{}_P))(\alpha), (f^{-1}(Ʊ^2{}_P))(ß)\}.$$

(iv) $(f^{-1}(Ʊ^2{}_P))(\alpha ß)) = Ʊ^2{}_P(f(\alpha ß))$

$$= Ʊ^2{}_P(f(\alpha - ß)$$
$$= Ʊ^2{}_P(f(\alpha), f(ß))$$
$$\leq max\{(Ʊ^2{}_P(f(\alpha)), Ʊ^2{}_P(f(ß))\}$$
$$= \max\{(f^{-1}(Ʊ^2{}_P))(\alpha), (f^{-1}(Ʊ^2{}_P))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_P))(\alpha ß) \leq \max\{(f^{-1}(Ʊ^2{}_P))(\alpha), (f^{-1}(Ʊ^2{}_P))(ß)\}.$$

Hence, f^{-1}(Ȟ) is an PMĂƑ subring of $Ř_1$.

Theorem 3.1.4. Suppose

Ř$_1$and Ř$_2$be two rings. Suppose f: Ř$_1$ → Ř$_2$is an anti-homomorphism onto rings. Suppose Ǥ = {(α, Ʊ$_M(\alpha)$, Ʊ$_N(\alpha)$)/$\alpha \epsilon$ Ř$_1$} be an PMĂƑ subring of Ř$_1$then f(Ǥ) is an ĬMĂƑ subring of Ř$_2$, if Ǥ has an inf property and Ǥ is f-invariant.

Proof: Suppose Ǥ = {(α, Ʊ$_M(\alpha)$, Ʊ$_N(\alpha)$)/$\alpha \epsilon$ Ř$_1$} be an ĬMĂƑ subring of Ř$_1$. Then, f(Ǥ) = {(ß, f(Ʊ$^2{}_M$(ß)), f(Ʊ$^2{}_N$(ß)))/ß ϵ Ř$_2$}. There exist α, ß ϵ Ř$_1$such that $f(\alpha)$, f(ß) ϵ Ř$_2$,

(i) $(f(Ʊ^2{}_M))(f(\alpha) - f(ß)) = (f(Ʊ^2{}_M))(f(ß - \alpha))$

$= Ʊ^2{}_M(ß - \alpha)$
$= Ʊ^2{}_M(\alpha - ß)$
$\leq \max\{ Ʊ^2{}_M(\alpha),\ Ʊ^2{}_M(ß)\}$
$= \max\{(f(Ʊ^2{}_M))(f(\alpha)),\ (f(Ʊ^2{}_M))(f(ß))\}$

$(f(Ʊ^2{}_M))(f(\alpha) - f(ß) \leq \max\{(f(Ʊ^2{}_M))(f(\alpha)),\ (f(Ʊ^2{}_M))(f(ß))\}$.

(ii) $(f(Ʊ^2{}_M))(f(\alpha)\, f(ß)) = (f(Ʊ^2{}_M))(f(ß\alpha))$

$= Ʊ^2{}_M(ß\alpha)$
$\leq max\{ Ʊ^2{}_M(\alpha), Ʊ^2{}_M(ß)\}$
$= max\{(f(Ʊ^2{}_M))(f(\alpha)), (f(Ʊ^2{}_M))(f(ß))\}$

$(f(Ʊ^2{}_M))f(\alpha)f(ß)) \leq max\{(f(Ʊ^2{}_M))(f(\alpha)), (f(Ʊ^2{}_M))(f(ß))\}$.

(iii) $(f(Ʊ^2{}_N))(f(\alpha) - f(ß)) = (f(Ʊ^2{}_N))(f(ß - \alpha))$

$= Ʊ^2{}_N(ß - \alpha)$
$\geq min\{ Ʊ^2{}_N(\alpha), Ʊ^2{}_N(ß)\}$
$= min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}$

$(f(Ʊ^2{}_N))(f(\alpha) - f(ß) \geq min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}$.

(iv) $(f(Ʊ^2{}_N))(f(\alpha)\, f(ß)) = (f(Ʊ^2{}_N))(f(ß\alpha))$

$= Ʊ^2{}_N(ß\alpha)$
$\geq min\{ Ʊ^2{}_N(\alpha), Ʊ^2{}_N(ß)\}$
$= min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}$

$(f(Ʊ^2{}_N))f(\alpha)f(ß)) \geq min\{(f(Ʊ^2{}_N))(f(\alpha)), (f(Ʊ^2{}_N))(f(ß))\}$.

Hence, f(Ǥ) is an PMĂƑ subring of Ř$_2$.

Theorem 3.1.5.

Suppose Ř$_1$and Ř$_2$be two rings. Suppose f: Ř$_1$ → Ř$_2$is an anti-homomorphism onto rings.

Suppose Ȟ = {(ß, Ʊ$_O$(ß), Ʊ$_P$(ß))/ß ϵ Ř$_2$} be an PMĂƑ subring of Ř$_2$then f^{-1}(Ȟ)is an PMĂƑ subring of Ř$_1$.

Proof

Ȟ = {(ß, Ʊ$_O$(ß), Ʊ$_P$(ß))/ß ϵ Ř$_2$} be an ĬMĂƑ subring of Ř$_2$.

Then, f^{-1}(Ȟ) = {(α, f^{-1}(Ʊ$^2{}_O(\alpha)$), f^{-1}(Ʊ$^2{}_P(\alpha)$))/ $\alpha \epsilon$ Ř$_1$}.

For any α, ß ϵ Ř$_1$, $f(\alpha)$, f(ß) ϵ Ř$_2$,

(i) $f^{-1}(Ʊ^2{}_O))(\alpha - ß)) = Ʊ^2{}_O(f(\alpha - ß))$

$= Ʊ^2{}_O(f(ß), f(\alpha))$
$= Ʊ^2{}_O(f(\alpha), f(ß)$
$\leq max\{(Ʊ^2{}_O(f(\alpha)),\quad Ʊ^2{}_O(f(ß))\}$
$= max\{ (f^{-1}(Ʊ^2{}_O))(\alpha),\quad (f^{-1}(Ʊ^2{}_O))(ß)\}$

$(f^{-1}(Ʊ^2{}_O))(\alpha - ß) \leq max\{(f^{-1}(Ʊ^2{}_O))(\alpha),\quad (f^{-1}(Ʊ^2{}_O))(ß)\}$.

Dialogue Social Science Review (DSSR)
www.thedssr.com

ISSN Online: 3007-3154
ISSN Print: 3007-3146

DIALOGUE SOCIAL SCIENCE REVIEW

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(ii) $(f^{-1}(Ʊ^2{}_O))(\alpha ß)) = Ʊ^2{}_O(f(\alpha ß))$

$$= Ʊ^2{}_O(f(ß), f(\alpha))$$
$$\leq min\{(Ʊ^2{}_O(f(\alpha)), Ʊ^2{}_O(f(ß))\}$$
$$= min\{(f^{-1}(Ʊ^2{}_O))(\alpha), \quad (f^{-1}(Ʊ^2{}_O))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_O))(\alpha ß) \leq min\{(f^{-1}(Ʊ^2{}_O))(\alpha), (f^{-1}(Ʊ^2{}_O))(ß)\}.$$

(iii) $f^{-1}(Ʊ^2{}_P))(\alpha - ß)) = Ʊ^2{}_P(f(\alpha - ß))$

$$= Ʊ^2{}_P(f(ß) - f(\alpha))$$
$$= Ʊ^2{}_P(f(\alpha) - f(ß))$$
$$\geq min\{(Ʊ^2{}_P(f(\alpha)), Ʊ^2{}_P(f(ß))\}$$
$$= min\{(f^{-1}(Ʊ^2{}_P))(\alpha), (f^{-1}(Ʊ^2{}_P))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_P))(\alpha - ß) \geq min\{(f^{-1}(Ʊ^2{}_P))(\alpha), (f^{-1}(Ʊ^2{}_P))(ß)\}.$$

(iv) $(f^{-1}(Ʊ^2{}_P))(\alpha ß)) = Ʊ^2{}_P(f(\alpha ß))$

$$= Ʊ^2{}_P(f(\alpha), f(ß))$$
$$\geq max\{(Ʊ^2{}_P(f(\alpha)), Ʊ^2{}_P(f(ß))\}$$
$$= max\{(f^{-1}(Ʊ^2{}_P))(\alpha), \quad (f^{-1}(Ʊ^2{}_P))(ß)\}$$
$$(f^{-1}(Ʊ^2{}_P))(\alpha ß) \geq max\{(f^{-1}(Ʊ^2{}_P))(\alpha), \quad (f^{-1}(Ʊ_P))(ß)\}.$$

Hence, f^{-1}(Ȟ) is an ĬMĄƑ subring of $Ř_1$.

Conclusion

This study exhibits the generalization of the framework of intuitionistic multi-anti fuzzy rings by proposing a significant and innovative extension of the notion homomorphism under Pythagorean multi anti fuzzy rings. As one of the basic idea, the homomorphism offers a mathematical foundation for systematically comparison of many Pythagorean multi-anti fuzzy rings. The formal definition of homomorphism under PMAFRs is given and using the extension principle the images and pre images are defined. It is observed that these constructions satisfy many useful characteristics which are listed and the detailed proofs are given.

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ISSN Online: 3007-3154
ISSN Print: 3007-3146

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