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Cartesian Product of the Pythagorean Multi Anti-Fuzzy Rings

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ABSTRACT
Pythagorean fuzzy set, an extension of both the fuzzy set as well as the intuitionistic fuzzy set, provides a flexible framework to deal with uncertainty and ambiguity. Ring theory is one of the crucial study within the modern algebra which helps in dealing with some complex problems of modern science and technology. Cartesian product is one of the key operations defined on rings which is further used to study mappings and the characterization of these maps. In this article we propose a novel form of Cartesian product constructed between Pythagorean multi anti fuzzy rings. We also discuss some of its characteristics and use this product to further study the image, pre image and some other properties of the homomorphism defined within this framework.

Introduction
Since its introduction by Lotfi A. Zadeh in 1965 (Zadeh, 1965), fuzzy set theory has become a groundbreaking idea in mathematics and has found useful applications in a number of domains, such as pattern recognition, artificial intelligence, decision-making, and control systems. Fuzzy set theory's primary attraction is its capacity to deal with ambiguity and uncertainty, which traditional set theory finds difficult to manage. The idea of intuitionistic fuzzy sets (ĬƑSs), first proposed by Atanassov in 1983 (Atanassov, 1986), expanded fuzzy set theory by adding two new parameters: the degree of membership and the degree of non-membership. By adding multi-dimensional membership functions. The ĬƑS were further generalized by Yager by the construction of the Pythagorean Fuzzy Sets (PFSs) (Yager, 2013). Sabu Sebastian developed multi-fuzzy set theory (Sabu, 2011), which expanded on these fundamental concepts and further generalized fuzzy sets. This development made it possible to model systems with several criteria or characteristics, which makes it especially applicable in fields like multi-criteria optimization and multi-objective decision-making. R. Muthuraj and S.
By proposing fuzzy subgroups, Rosenfeld (Rosenfeld, 1971) expanded fuzzy set theory to algebraic structures. WJ Liu proposed the ring structure and its ideals under the fuzzy

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framework (Liu, 1982) which is further studied by well-known researchers in the contexts of multi fuzzy (Al-Husban, 2022), intuitionistic fuzzy (Marashdeh, 2001) and Pythagorean fuzzy settings (Razaq, 2023). Intuitionistic multi fuzzy near-rings and ideals associated with it were studied by Batool et. al. (Batool, 2023). Balamurugan (Muthuraj R. ,., 2013) developed the idea of multi-anti fuzzy subgroups, which integrate anti-membership functions into multi-fuzzy set theory, as an extension of these ideas. Recently the characterization of intuitionistic multi anti-fuzzy was done by Muthuraj (Muthuraj, 2022) in which the Cartesian product and homomorphisms were focused.

The work specifically focusses on defining and analyzing the basic features of the Cartesian product of an intuitionistic multi-anti fuzzy ring of a given ring. In addition, we investigate the characteristics of intuitionistic multi-anti fuzzy rings' pictures under homomorphism and anti-homomorphism, emphasizing the importance of these aspects in comprehending the structural connections among these fuzzy systems. Cartesian product is a basic mathematical procedure for combining two or more sets into a new composite structure. By joining many intuitionistic multi-anti fuzzy sets, the Cartesian product may be used to built increasingly intricate fuzzy systems in the setting of intuitionistic multi-anti fuzzy rings.

Preliminaries

This section is devoted to describe some of the pre-requisite notions which are used in this study and without having an acquaintance of these concepts it will be difficult to understand the text.

Definition 2.0.1. Suppose a non-empty set E then the fuzzy subset (FSS) M of E can be defined as

$$M = \{(\alpha,\ \mho(\alpha)) : \alpha \in E \text{ and } \mho(\alpha) : E \to [0, 1]\}.$$

Example 2.0.2.

We can define a FSS N for the set $X = \{1,\ 2,\ 3,\ 4,\ 5\}$ as

$$N = \{(1,\ .5), (2,\ .2), (3,\ .4), (4,\ .3), (5,\ .1)\}$$

If $X = \{15,\ 25,\ 35,\ 45,\ 55\}$ shows weights then the FSs "Heavy Weight" and "Light Weight" will b

$$\text{Light Weight} = \{(15,\ .9)(25,\ .8)(35,\ .5)(45,\ 1)(55,\ 0)\}$$
$$\text{Heavy Weight} = \{(15,\ 0)(25,\ .5)(35,\ .8)(45,\ 1)(55,\ 1)\}$$

Definition 2.0.3. Suppose A is a set then the ĬƑS described on A can be represented as $M = \{(\alpha,\ \mho(\alpha),\ \Omega(\alpha)) : \alpha \in M,\ \mho(\alpha) : A \to [0,\ 1] \text{ and } \Omega(\alpha) : \to [0,\ 1]\}$ such that $0 \le \mho(\alpha) + \Omega(\alpha) \le 1$.

Definition 2.0.4. The Pythagorean fuzzy set M constructed on the discourse set A can be represented as $M = \{(\alpha,\ \mho(\alpha),\ \Omega(\alpha)) : \alpha \in M, \mho(\alpha) : A \to [0,\ 1] \text{ and } \Omega(\alpha) : \to [0,\ 1]\}$ Such that $0 \le {\mho_1}^2(\alpha) + {\Omega_1}^2(\alpha) \le 1$

Example 2.0.5 For the ground set $A = \{1,\ 2,\ 3\}$ the set below describes the PFSs

$$M = \{(1,\ 0.2,\ 0.9), (2,\ 0.3,\ 0.4), (3,\ 0.8,\ 0.2)\}$$

Definition 2.0.6. Suppose a non- empty set X. A multi-fuzzy set M is defined as

$$M = \{(\alpha,\ \ \mho_1(\alpha),\ \mho_2(\alpha),\ \ \mho_3(\alpha),\ \ \mho_4(\alpha), \ldots\ldots) : \alpha \in X\},$$

where, $\mho_\iota(\alpha) \to [0,\ 1]$ for all ι. For instance, set written below is multi fuzzy

$$M = \{(a, \{.1, \ .2, \ .4\}, (b, \{.3, \ .4, \ .001\}), (c, \ \{.2, \ .1, \ .25\})\}$$

Definition 2.0.7. An Intuitionistic multi-fuzzy set M is defined as $M = \{(\alpha, \mho_1(\alpha), \ \mho_2(\alpha), \ \mho_3(\alpha), \ \ldots, \mho_k; \ \Omega_1(\alpha), \ \Omega_2(\alpha), \ \Omega_3(\alpha), \ldots, \ \Omega_k(\alpha)) : \alpha \in \ M\}$ where, $\mho_k(\alpha) \to \ [0, \ 1]$ for all k and $\Omega_k(\alpha) \to \ [0, \ 1]$.

Definition 2.0.8. Take X to be the set of discourse then one can describe the Pythagorean multi fuzzy set

$$P_M^F = \{(\alpha, \mathcal{C}\Omega(\alpha), \ \mathcal{C}\mho(\alpha)) \ / \ \alpha \in \ X\}$$

is described by two maps $\mathcal{C}\Omega(\alpha): X \to Q$ and $\mathcal{C}\mho(\alpha): X \to Q$, labeling as functions of count membership and count non-membership, where the set Q exhibits family of all possible crisp multi-sets produced from the interval [0,1] and for each $\alpha \in \ X$, $\mathcal{C}\Omega(\alpha)$ will be a decreasingly ordered progression such as $(\mho_1(\alpha))^2 \ \geq \ (\mho_2(\alpha))^2 \geq \ (\mho_3(\alpha))^2 \geq \ \ldots \geq \ (\mho_k(\alpha))^2$ and $\mathcal{C}\mho(\alpha)$ could be of any order $(\Omega_1(\alpha))^2, \ (\Omega_2(\alpha))^2, \ (\Omega_3(\alpha))^2, \ldots, \ (\Omega_k(\alpha))^2$. For each $\alpha \in \ X, 0 \leq (\mathcal{C}\Omega(\alpha))^2 + (\mathcal{C}\mho(\alpha))^2 \ \leq 1$.

Example 2.0.9. Let $M = \{\sigma, \rho, \eta\}$ then $\mathcal{A}$ is intuitionistic fuzzy multi-set over $\mathcal{N}$ with count functions:

$$\mathcal{C}\Omega(\alpha) = \begin{cases} 0.3, 0.3, 0.5 & if \ \alpha = \sigma \\ 0.02, 0.02, 0.02 & if \ \alpha = \rho \\ 1, 1, 0.6 & if \ \alpha = \eta \end{cases}$$

$$\mathcal{C}\mho(\alpha) = \begin{cases} 0.5, 0.5, 0.3 & if \ \alpha = \sigma \\ 0.8, 0.8, 0.8 & if \ \alpha = \rho \\ 0.4 & if \ \alpha = \eta \end{cases}$$

Definition 2.0.10. Given two Pythagorean fuzzy multi-sets A and B over X with maps $\mathcal{C}\Omega_A(\alpha)$ and $\mathcal{C}\Omega_B(\alpha)$ labeling as functions of count membership of A and B and $\mathcal{C}\mho_A(\alpha)$ and $\mathcal{C}\mho_B(\alpha)$ showcases count non-membership, then:

i. $A \subseteq B$ if $(\mathcal{C}\Omega_A(\alpha))^2 \leq (\mathcal{C}\Omega_B(\alpha))^2$ and $(\mathcal{C}\mho_A(\alpha))^2 \leq (\mathcal{C}\mho_B(\alpha))^2 \ \forall \ \alpha \in \ X$

ii. $A = B$ if $(\mathcal{C}\Omega_A(\alpha))^2 = \ (\mathcal{C}\Omega_B(\alpha))^2$ and $(\mathcal{C}\mho_A(\alpha))^2 = (\mathcal{C}\mho_B(\alpha))^2 \ \forall \ \alpha \in \ X$.

iii. $\mathcal{C}\Omega_{A\cap B}(\alpha) = \min\left\{(\mathcal{C}\Omega_A(\alpha))^2, (\mathcal{C}\Omega_B(\alpha))^2\right\}$ and $\mathcal{C}\mho_{A\cap B}(\alpha) = \max\{(\mathcal{C}\mho_A(\alpha))^2, (\mathcal{C}\mho_B(\alpha))^2\}$.

iv. $\mathcal{C}\Omega_{A\cup B}(\alpha) = \max\left\{(\mathcal{C}\Omega_A(\alpha))^2, (\mathcal{C}\Omega_B(\alpha))^2\right\}$ and
$\mathcal{C}\mho_{A\cup B}(\alpha) = \min\left\{(\mathcal{C}\mho_A(\alpha))^2, (\mathcal{C}\mho_B(\alpha))^2\right\}$.

v. Complement of an intuitionistic fuzzy multi-set is defined as;

$$A^c = \{< \alpha, (\mathcal{C}\mho_A(\alpha))^2, (\mathcal{C}\Omega_A(\alpha))^2 > / \ \alpha \in \ X\}$$

Definition 2.0.11. Let M be a fuzzy set on a ring Ř. Then M is a fuzzy subring of Ǥ if and only if

i. $\mho_M(\alpha - ß) \geq \min\{ \mho_M(\alpha), \Omega_M(ß)\}$

ii. $\mho_M(\alpha . ß) \geq \min\{ \mho_M(\alpha), \Omega_M(ß)\}$

Definition 2.0.12. Suppose a ring Ř, and $M = \{(\alpha, \Omega_M(\alpha), \ \mho_N(\alpha)) | \alpha \in Ř \}$ be Pythagorean fuzzy ring over Ř, if the given conditions are holds,

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i. $\left(\Omega_M(\alpha - ß)\right)^2 \geq \min\{(\Omega_M(\alpha))^2, (\Omega_M(ß))^2\}$

ii. $\left(\Omega_M(\alpha ß)\right)^2 \geq \min\left\{(\Omega_M(\alpha))^2, (\Omega_M(ß))^2\right\}$

iii. $\left(\mho_N(\alpha - ß)\right)^2 \leq \max\left\{(\mho_N(\alpha))^2, ((\mho_N(ß))^2\right\}$

iv. $\left(\mho_N(\alpha ß)\right)^2 \leq \max\left\{(\mho_N(\alpha))^2, ((\mho_N(ß))^2\right\}, \quad \forall\ \alpha, ß \in Ř$

Definition 2.0.13. Suppose a ring Ř, and $M = \{(\alpha, \Omega_M(\alpha), \mho_N(\alpha)) | \alpha \in Ř\}$ be Pythagorean fuzzy ring over Ř, if the given conditions are holds,

i. $\left(\Omega_M(\alpha - ß)\right)^2 \leq \max\{(\Omega_M(\alpha))^2, (\Omega_M(ß))^2\}$

ii. $\left(\Omega_M(\alpha ß)\right)^2 \leq \max\left\{(\Omega_M(\alpha))^2, (\Omega_M(ß))^2\right\}$

iii. $\left(\mho_N(\alpha - ß)\right)^2 \geq \min\left\{(\mho_N(\alpha))^2, ((\mho_N(ß))^2\right\}$

iv. $\left(\mho_N(\alpha ß)\right)^2 \geq \min\left\{(\mho_N(\alpha))^2, ((\mho_N(ß))^2\right\}, \quad \forall\ \alpha, ß \in Ř$

Main Results

Here we describe the novel conception of the Cartesian product under the framework of Pythagorean multi anti-fuzzy rings and exhibit some interesting algebraic characteristics.

Definition 3.1.1.

Consider a ring Ř and $Ɠ = \{(\alpha, \mathcal{C}\Omega_M(\alpha), \mathcal{C}\mho_V(\alpha)) | \alpha \in Ř\}$ as an PMĄ̆ƑS over a ring Ř occupied with count membership $\mathcal{C}\Omega_M(\alpha)$ and count non-membership $\mathcal{C}\mho_N(\alpha)$. Then Ɠ is called an PMĄ̆ƑŘ over Ř if and only if ∀ α, ß ϵ Ř the following conditions are hold.

i. $\left(\mathcal{C}\Omega_M(\alpha - ß)\right)^2 \leq \max\{(\mathcal{C}\Omega_M(\alpha))^2, (\mathcal{C}\Omega_M(ß))^2\}$

ii. $\left(\mathcal{C}\Omega_M(\alpha ß)\right)^2 \leq \max\left\{(\mathcal{C}\Omega_M(\alpha))^2, (\mathcal{C}\Omega_M(ß))^2\right\}$

iii. $\left(\mathcal{C}\mho_N(\alpha - ß)\right)^2 \geq \min\left\{(\mathcal{C}\mho_N(\alpha))^2, ((\mathcal{C}\mho_N(ß))^2\right\}$

iv. $\left(\mathcal{C}\mho_N(\alpha ß)\right)^2 \geq \min\left\{(\mathcal{C}\mho_N(\alpha))^2, ((\mathcal{C}\mho_N(ß))^2\right\}, \quad \forall\ \alpha, ß \in Ř$

Proposition 3.1.2. If we assume $Ɠ = \{(\alpha, \mathcal{C}\Omega_M(\alpha), \mathcal{C}\mho_N(\alpha)) | \alpha \in Ř\}$ as a PMĄ̆ƑŘ over a ring Ř. Clearly, ∀ α, ß ϵ Ř,

i. $\left(\mathcal{C}\Omega_M(\alpha)\right)^2 \geq \left(\mathcal{C}\Omega_M(0)\right)^2$ and $\left(\mathcal{C}\Omega_M(\alpha)\right)^2 = \left(\mathcal{C}\Omega_M(-\alpha)\right)^2$

ii. $\left(\mathcal{C}\Omega_M(\alpha - ß)\right)^2 = 0$ implies that $\left(\mathcal{C}\Omega_M(\alpha)\right)^2 = \left(\mathcal{C}\Omega_M(ß)\right)^2$

iii. $\left(\mathcal{C}\mho_N(\alpha)\right)^2 \leq \left(\mathcal{C}\mho_N(0)\right)^2$ and $\left(\mathcal{C}\mho_N(\alpha)\right)^2 = \left(\mathcal{C}\mho_N(-\alpha)\right)^2$

iv. $\left(\mathcal{C}\mho_N(\alpha - ß)\right)^2 = 0$ implies that $\left(\mathcal{C}\mho_N(\alpha)\right)^2 = \left((\mathcal{C}\mho_N(ß)\right)^2$

Example 3.1.3.

Consider (Z, +, .) is ring of integers. Suppose $Ɠ = \{(\alpha, \mathcal{C}\Omega_M(\alpha), \mathcal{C}\mho_N(\alpha)) / \alpha \in Ř\}$ is an ĬMĄ̆Ƒ over a ring Ř, where

$$\mathcal{C}\Omega_M(\alpha) = (\mathcal{C}\Omega_{M1}(\alpha), \mathcal{C}\Omega_{M2}(\alpha)) = \begin{cases} (0.2, \ 0.3) \text{ if } \alpha = 0 \\ (0.2, \ 0.1) \text{ if } \alpha \neq 0 \end{cases}$$

and

$$\mathcal{C}\mho_N(\alpha) = (\mathcal{C}\mho_{N1}(\alpha), \mathcal{C}\mho_{N2}(\alpha)) = \begin{cases} (0.8, \ 0.6) \text{ if } \alpha = 0 \\ (0.7, \ 0.9) \text{ if } \alpha \neq 0 \end{cases}$$

Evidently, Ɠ is an PMĄ̆ƑŘ of dimension 2.

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Definition 3.1.4.

Suppose $Ǥ = \{(\alpha, \mathcal{C}\Omega_M(\alpha), \mathcal{C}Ʊ_N(\alpha)) / \alpha \in Ř_1\}$ and $Ȟ = \{(ß, \mathcal{C}\Omega_O(ß), \mathcal{C}Ʊ_P(ß)) / ß \in Ř_2\}$ are PMĄ̆ƑŘ on rings $Ř_1$ and $Ř_2$, respectively. Then we define the anti-Cartesian product of Ǥ and Ȟ.

$$Ǥ \times Ȟ = \{((\alpha, ß), (\mathcal{C}\Omega_M \cup \mathcal{C}\Omega_O)(\alpha, ß), (\mathcal{C}Ʊ_N \cap \mathcal{C}Ʊ_P)(\alpha, ß)) / (\alpha, ß) \in Ř_1 \times Ř_2\},$$

Where

$$(\mathcal{C}\Omega_M \cup \mathcal{C}\Omega_P)(\alpha, ß) = \max\left\{(\mathcal{C}\Omega_M(\alpha))^2, (\mathcal{C}\Omega_O(ß))^2\right\}, \forall\ (\alpha, ß) \in Ř_1 \times Ř_2$$

$$(\mathcal{C}Ʊ_N \cap \mathcal{C}Ʊ_Q)(\alpha, ß) = \min\left\{(\mathcal{C}Ʊ_N(\alpha))^2, ((\mathcal{C}Ʊ_P(ß))^2\right\}, \forall\ (\alpha, ß) \in Ř_1 \times Ř_2$$

Theorem 3.1.5.

Suppose $Ǥ = \{(\alpha, \mathcal{C}\Omega_M(\alpha), \mathcal{C}Ʊ_N(\alpha)) / \alpha \in Ř_1\}$ and $Ȟ = \{(ß, \mathcal{C}\Omega_O(ß), \mathcal{C}Ʊ_P(ß)) / ß \in Ř_2\}$ are PMĄ̆ƑŘ on rings $Ř_1$ and $Ř_2$, respectively. Hence, $Ǥ \times Ȟ$ is an PMĄ̆Ƒ subring of the ring $Ř_1 \times Ř_2$.

Proof.

We are using a shorthand $Ʊ_{MUO}$ for $\mathcal{C}\Omega_M \cup \mathcal{C}\Omega_O$ and $Ʊ_{N\cap P}$ to denote $\mathcal{C}Ʊ_N \cap \mathcal{C}Ʊ_P$. Then

$$Ǥ \times Ȟ = \{((\alpha, ß), Ʊ_{MUO}(\alpha, ß), Ʊ_{N\cap P}(\alpha, ß)) / (\alpha, ß) \in Ř_1 \text{ and } Ř_2\}$$

Suppose $\alpha, ß \in Ř_1 \times Ř_2$, where $\alpha = (l,\ m)$, $ß = (t,\ v)$,

i. $$\begin{aligned} Ʊ^2{}_{MUO}(\alpha, ß) &= \max\{(Ʊ_M(l-t))^2, (Ʊ_O(m-v))^2\} \\ &\le \operatorname{Max}\{\max\{Ʊ_M^2(l), Ʊ^2{}_M(t)\}, \max\{Ʊ_O^2(m), Ʊ_O^2(v)\}\} \\ &= \max\{\max\{Ʊ_M^2(l), Ʊ_O^2(m)\}, \max\{Ʊ_M^2(t), Ʊ_O^2(v)\}\} \\ &= \max\{Ʊ_{MUO}^2(l, m), Ʊ^2{}_{MUO}(t, v)\} \end{aligned}$$
$$Ʊ^2{}_{MUO}(\alpha - ß) \le \max\{Ʊ_{MUO}^2(\alpha), Ʊ^2{}_{MUO}(ß)\}$$

ii. $$\begin{aligned} Ʊ^2{}_{MUO}(\alpha ß) &= Ʊ^2{}_{MUO}((l, m).(t, v)) = Ʊ_{MUO}^2(lt, mv) \\ &= \max\{Ʊ^2{}_M(lt), Ʊ^2{}_O(mv)\} \\ &\ge \operatorname{Max}\{\max\{Ʊ^2{}_M(l), Ʊ^2{}_M(t)\}, \max\{Ʊ^2{}_O(m), Ʊ^2{}_O(v)\}\} \\ &= \max\{\max\{Ʊ^2{}_M(l), Ʊ^2{}_O(m)\}, \max\{Ʊ^2{}_M(t), Ʊ^2{}_O(v)\}\} \end{aligned}$$
$$Ʊ^2{}_{MUO}(\alpha ß) \le \max\{Ʊ^2{}_{MUO}(\alpha), Ʊ^2{}_{MUO}(ß)\}.$$

iii. $$\begin{aligned} Ʊ^2{}_{N\cap P}(\alpha, ß) &= \min\{Ʊ_N^2(l-t), Ʊ_P^2(m-v)\} \\ &\ge \min\{\min\{Ʊ^2{}_N(l), Ʊ^2{}_N(t)\}, \min\{Ʊ^2{}_P(m), Ʊ^2{}_P(v)\}\} \\ &= \min\{\min\{Ʊ^2{}_N(l), Ʊ^2{}_P(m)\}, \min\{Ʊ^2{}_N(t), Ʊ^2{}_P(v)\}\} \\ &= \min\{Ʊ^2{}_{N\cap P}(l,\ m), Ʊ^2{}_{N\cap P}(t,\ v)\} \end{aligned}$$
$$Ʊ^2{}_{N\cap P}(\alpha - ß) \ge \min\{Ʊ^2{}_{N\cap P}(\alpha), Ʊ^2{}_{N\cap P}(ß)\}.$$

iv. $$\begin{aligned} Ʊ_{N\cap P}^2(\alpha ß) &= Ʊ_{N\cap P}^2((l,\ m).(t,\ v)) = Ʊ_{N\cap P}^2(lt, mv) \\ &= \min\{Ʊ^2{}_N(l, t), Ʊ^2{}_P(m, v)\} \\ &\ge \operatorname{Min}\{\min\{Ʊ^2{}_N(l), Ʊ^2{}_N(t)\}, \min\{Ʊ^2{}_P(m), Ʊ^2{}_P(v)\}\} \\ &= \min\{\min\{Ʊ^2{}_N(l), Ʊ^2{}_P(m)\}, \min\{Ʊ^2{}_N(t), Ʊ^2{}_P(v)\}\} \\ &= \min\{Ʊ^2{}_{N\cap P}(l, m), Ʊ^2{}_{N\cap P}(t, v)\} \end{aligned}$$
$$Ʊ^2{}_{N\cap P}(\alpha, ß) \ge \min\{Ʊ^2{}_{N\cap P}(\alpha), Ʊ^2{}_{N\cap P}(ß)\}.$$

This implies that $Ǥ \times Ȟ$ is also an ĬMĄ̆ƑŘ on rings $Ř_1 \times Ř_2$.

Theorem 3.1.6

Suppose $Ǥ = \{(\alpha,\ Ʊ_M(\alpha),\ Ʊ_N(\alpha))/\alpha \in Ř_1\}$ and $Ȟ = \{(ß,\ Ʊ_O(ß),\ Ʊ_P(ß))/ß \in Ř_2\}$ are ĬMĄ̆ƑS on a ring $Ř_1$ and $Ř_2$ correspondingly. Let $Ř_1$ and $Ř_2$ have identity elements that

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is 0_1 and 0_2 respectively. If an ĬMĄĂƑS of $\check{R}_1$ and $\check{R}_2$ is the anti-Cartesian product Ǥ × Ȟ, then at least one of the following conditions must be true.

i. $Ʊ^2_O(0_2) \leq Ʊ^2{}_M(\alpha)$ and $Ʊ^2{}_P(0_2) \geq Ʊ^2{}_{\mathbf{N}}(\alpha), \forall\ \alpha \in \check{R}_1$

ii. $Ʊ_M(0_1) \leq Ʊ_O(ß)$ and $Ʊ_{\mathbf{N}}(0_1) \geq Ʊ_P(ß), \forall\ ß \in \check{R}_2$.

Proof:

Suppose $Ǥ = \{(\alpha,\ Ʊ_M(\alpha), Ʊ_{\mathbf{N}}(\alpha))/\alpha \in \check{R}_1\}$ and $Ȟ = \{(ß,\ Ʊ_O(ß),\ Ʊ_P(ß))/ß \in \check{R}_2\}$ are ĬMĄĂƑS on a ring $\check{R}_1$ and $\check{R}_2$ correspondingly.

Then, $Ǥ \times Ȟ = \{\big((\alpha, ß),\ Ʊ^2{}_{M\cup O}(\alpha, ß),\ Ʊ^2{}_{N\cap P}(\alpha, ß)\big): (\alpha, ß) \in \check{R}_1 \times \check{R}_2\}$,

Where, $Ʊ^2{}_{M\cup O}(\alpha, ß) = \max\{Ʊ^2{}_M(\alpha),\ Ʊ^2{}_O(ß)\}, \forall\ (\alpha, ß) \in \check{R}_1 \times \check{R}_2$,

$Ʊ^2{}_{N\cap P}(\alpha, ß) = \min\{Ʊ^2{}_{\mathbf{N}}(\alpha), Ʊ^2{}_P(ß)\}, \forall\ (\alpha, ß) \in \check{R}_1 \times \check{R}_2$.

Consider $(\alpha, ß) \in \check{R}_1 \times \check{R}_2$, where $\alpha = (l,\ m)$, $ß = (t,\ v)$,

Suppose an ĬMĄĂƑŘ of $\check{R}_1 \times \check{R}_2$ is the anti-Cartesian product Ǥ × Ȟ. By contraposition, suppose that the given condition (i), (ii) is not hold then we can interpret $\alpha \in \check{R}_1$ and $ß \in \check{R}_2$ such that

i. and $Ʊ^2{}_P(0_2) \leq Ʊ^2{}_{\mathbf{N}}(\alpha), \forall\ \alpha \in \check{R}_1$

ii. and $Ʊ^2{}_{\mathbf{N}}(0_1) \leq Ʊ^2{}_P(ß), \forall\ ß \in \check{R}_2$

We have,

$$\begin{aligned} Ʊ^2{}_{M\cup O}(\alpha,\ ß) &= \max\{Ʊ^2{}_M(\alpha),\ Ʊ^2{}_O(ß)\} \\ &< \max\{Ʊ^2{}_O(0_2),\ Ʊ^2{}_M(0_1)\} \\ &= \max\{Ʊ^2{}_M(0_1),\ Ʊ^2{}_O(0_2)\} \\ &= Ʊ^2{}_{M\cup O}(0_1, 0_2) \end{aligned}$$

$$Ʊ^2{}_{M\cup O}(\alpha, ß) < Ʊ^2{}_{M\cup O}(0_1, 0_2)$$

Also,

$$\begin{aligned} Ʊ^2{}_{N\cap P}(\alpha, ß) &= \min\{Ʊ^2{}_{\mathbf{N}}(\alpha),\ Ʊ^2{}_P(ß)\} \\ &> \min\{Ʊ^2{}_P(0_2), Ʊ^2{}_{\mathbf{N}}(0_1)\} \\ &= \min\{Ʊ^2{}_{\mathbf{N}}(0_1),\ Ʊ^2{}_P(0_2)\} \\ &= Ʊ^2{}_{N\cap P}(0_1, 0_2) \end{aligned}$$

$$Ʊ^2{}_{N\cap P}(\alpha, ß) > Ʊ^2{}_{N\cap P}(0_1, 0_2).$$

Which implies that Ǥ × Ȟ is not an ĬMĄĂƑ subring of $\check{R}_1 \times \check{R}_2$.

Hence, either $Ʊ^2{}_O(0_2) \leq Ʊ^2{}_M(\alpha)$ and $Ʊ^2{}_P(0_2) \geq Ʊ^2{}_{\mathbf{N}}(\alpha), \forall\ \alpha \in \check{R}_1$

or $Ʊ^2{}_M(0_1) \leq Ʊ^2{}_O(ß)$ and $Ʊ^2{}_{\mathbf{N}}(0_1) \geq Ʊ^2{}_P(ß), \forall\ ß \in \check{R}_2$.

Theorem 3.1.7

Suppose $Ǥ = \{(\alpha,\ Ʊ_M(\alpha),\ Ʊ_{\mathbf{N}}(\alpha))/\alpha \in \check{R}_1\}$ and $Ȟ = \{(ß,\ Ʊ_O(ß),\ Ʊ_P(ß))/ß \in \check{R}_2\}$ are ĬMĄĂƑS on ring $\check{R}_1$ and $\check{R}_2$ correspondingly, such that $Ʊ^2{}_O(0_2) \leq Ʊ^2{}_M(\alpha)$ and $Ʊ^2{}_P(0_2) \geq Ʊ^2{}_{\mathbf{N}}(\alpha),\ \forall\ \alpha \in \check{R}_1$, where 0_2 is the identity element of $\check{R}_2$. Suppose an ĬMĄĂƑ subring of $\check{R}_1 \times \check{R}_2$. is the anti-Cartesian product Ǥ × Ȟ, then Ǥ is an intuitionistic multi-anti-fuzzy subring of $\check{R}_1$.

Proof:

Suppose $Ǥ = \{(\alpha,\ Ʊ_M(\alpha), Ʊ_{\mathbf{N}}(\alpha))/\alpha \in \check{R}_1\}$ and $Ȟ = \{(ß,\ Ʊ_O(ß),\ Ʊ_P(ß))/ß \in \check{R}_2\}$ are ĬMĄĂƑS on a ring $\check{R}_1$ and $\check{R}_2$ correspondingly.

Then, $Ǥ \times Ȟ = \{((\alpha, ß),\ Ʊ^2{}_{M\cup O}(\alpha, ß),\ Ʊ^2{}_{N\cap P}(\alpha, ß))/\ (\alpha, ß) \in \check{R}_1 \times \check{R}_2\}$,

Where, $Ʊ^2{}_{M\cup O}(\alpha, ß) = \max\{Ʊ^2{}_M(\alpha),\ Ʊ^2{}_O(ß)\}\ \forall\ (\alpha,\ ß) \in \check{R}_1 \times \check{R}_2$,

$Ʊ^2{}_{N\cap P}(\alpha, ß) = \min\{Ʊ^2{}_{\mathbf{N}}(\alpha), Ʊ^2{}_P(ß)\}, \forall\ (\alpha, ß) \in \check{R}_1 \times \check{R}_2$.

If the anti-Cartesian product is an ĬMĄĂƑ subring of $\check{R}_1 \times \check{R}_2$ then Ǥ is

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an ĬMĂƑŘ of $Ř_1$ then

$$(α, 0_2), \quad (ß, 0_2) \in Ř_1 \times Ř_2.$$

Since, $Ʊ^2{}_O(0_2) \le Ʊ^2{}_M(α)$ and $Ʊ^2{}_P(0_2) \ge Ʊ^2{}_N(α)$, $\forall\ α \in Ř_1$, where 0_2 is identity element of $Ř_2$.

i. $$\begin{aligned} Ʊ^2{}_M(α - ß) &= \max\{Ʊ^2{}_M(α - ß), \quad Ʊ^2{}_O(0_2 - 0_2)\} \\ &= Ʊ^2{}_{MUO}(α - ß, 0_2 - 0_2) \\ &= Ʊ^2{}_{MUO}\big((α, 0_2) - (ß, 0_2)\big) \\ &\le \max\{Ʊ^2{}_{MUO}(α - 0_2), Ʊ^2{}_{MUO}(ß - 0_2)\} \\ &= \max\{\max\{Ʊ^2{}_M(α), Ʊ^2{}_O(0_2)\}, \max\{Ʊ^2{}_M(ß), Ʊ^2{}_O(0_2)\}\} \\ &= \max\{Ʊ^2{}_M(α), Ʊ^2{}_M(ß)\} \end{aligned}$$

$$Ʊ^2{}_M(α - ß) \le \max\{Ʊ^2{}_M(α), Ʊ^2{}_M(ß)\}$$

ii. $$\begin{aligned} Ʊ^2{}_O(αß) &= \max\{Ʊ^2{}_M(0_1 0_1), Ʊ^2{}_O(αß)\} \\ &= Ʊ^2{}_{MUO}(0_1 0_1, αß) \\ &= Ʊ^2{}_{MUO}((0_1, α).(0_1, ß)) \\ &\le \max\{Ʊ^2{}_{MUO}((0_1, α), Ʊ^2{}_{MUO}(0_1, ß)\} \\ &= \max\{\max\{Ʊ^2{}_M(0_1), Ʊ^2{}_O(α)\}, \max\{Ʊ^2{}_M(0_1), Ʊ^2{}_O(ß)\}\} \\ &= \max\{Ʊ^2{}_O(α), Ʊ^2{}_O(ß)\} \end{aligned}$$

$$Ʊ^2{}_O(αß) \le \max\{Ʊ^2{}_O(α), Ʊ^2{}_O(ß)\}$$

iii. $$\begin{aligned} Ʊ^2{}_{\mathbf{N}}(α - ß) &= \min\{Ʊ^2{}_{\mathbf{N}}(α - ß), Ʊ^2{}_P(0_2 - 0_2)\} \\ &= Ʊ^2{}_{N\cap P}(α - ß, 0_2 - 0_2) \\ &= Ʊ^2{}_{N\cap P}((α, 0_2) - (ß, 0_2)) \\ &\ge \min\{(Ʊ^2{}_{N\cap P}(α, 0_2), Ʊ^2{}_{N\cap P}(ß, 0_2)\} \\ &= \min\{\min\{(Ʊ^2{}_{\mathbf{N}}(α), Ʊ^2{}_P(0_2)\}, \min\{Ʊ^2{}_{\mathbf{N}}(ß), Ʊ^2{}_P(0_2)\}\} \\ &= \min\{Ʊ^2{}_{\mathbf{N}}(α), Ʊ^2{}_{\mathbf{N}}(ß)\} \end{aligned}$$

$$Ʊ^2{}_{\mathbf{N}}(α - ß) \ge \min\{Ʊ^2{}_{\mathbf{N}}(α), Ʊ^2{}_{\mathbf{N}}(ß)\}$$

iv. $$\begin{aligned} Ʊ^2{}_{\mathbf{N}}(αß) &= \min\{Ʊ^2{}_{\mathbf{N}}(αß), Ʊ^2{}_P(0_2 0_2)\} \\ &= Ʊ^2{}_{N\cap P}(αß, 0_2 0_2) \\ &= Ʊ^2{}_{N\cap P}((α, 0_2)(ß, 0_2)) \\ &\ge \min\{Ʊ^2{}_{N\cap P}((α, 0_2), Ʊ^2{}_{N\cap P}((ß, 02)\} \\ &= \min\{\min\{Ʊ^2{}_{\mathbf{N}}(α), Ʊ^2{}_P(0_2)\}, \min\{Ʊ^2{}_{\mathbf{N}}(ß), Ʊ^2{}_P(0_2)\}\} \\ &= \min\{Ʊ^2{}_{\mathbf{N}}(α), Ʊ^2{}_{\mathbf{N}}(ß)\} \end{aligned}$$

$$Ʊ^2{}_{\mathbf{N}}(αß) \ge \min\{Ʊ^2{}_{\mathbf{N}}(α), Ʊ^2{}_{\mathbf{N}}(ß)\}$$

Hence, Ǥ is an ĬMĂƑ subring of $Ř_1$.

Theorem 3.1.8

Suppose $Ǥ = \{(α, Ʊ_M(α), Ʊ_N(α))/α \in Ř_1\}$ and $Ȟ = \{(ß, Ʊ_O(ß), Ʊ_P(ß))/ß \in Ř_2\}$ are ĬMĂƑS on a ring $Ř_1$ and $Ř_2$ correspondingly, such that $Ʊ^2{}_M(0_1) \le Ʊ^2{}_O(ß)$ and $Ʊ^2{}_N(0_1) \ge Ʊ^2{}_P(ß)$, $\forall\ ß \in Ř_2$, where 0_1 is identity element of $Ř_1$. If anti-Cartesian product $Ǥ \times Ȟ$ is an ĬMĂƑ subring of $Ř_1 \times Ř_2$, then H is an ĬMĂƑ subring of $Ř_2$.

Proof

Suppose $Ǥ = \{(α, Ʊ_M(α), Ʊ_N(α))/α \in Ř_1\}$ and $Ȟ = \{(ß, Ʊ_O(ß), Ʊ_P(ß))/ß \in Ř_2\}$ are ĬMĂƑS on a ring $Ř_1$ and $Ř_2$ correspondingly.

Then, $Ǥ \times Ȟ = \{((α, ß), Ʊ^2{}_{MUO}(α, ß), Ʊ^2{}_{N\cap P}(α, ß))/ (α, ß) \in Ř_1 \times Ř_2\}$,

Where, $$Ʊ^2{}_{MUO}(α, ß) = \max\{Ʊ^2{}_M(α), Ʊ^2{}_O(ß)\}, \forall\ (α, ß) \in Ř_1 \times Ř_2,$$

$$Ʊ^2{}_{N\cap P}(α, ß) = \min\{Ʊ^2{}_N(α), Ʊ^2{}_P(ß)\}, \forall\ (α, ß) \in Ř_1 \times Ř_2.$$

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Suppose $Ɠ \times Ȟ$ is an ĬMĂƑ subring of $Ř_1 \times Ř_2$ and $\alpha, ß \in Ř_2$ then $(0_2, \alpha), (0_2, ß) \in Ř_1 \times Ř_2$.
Since, $Ʊ^2{}_M(0_1) \le Ʊ^2{}_O(ß)$ and $Ʊ^2{}_N(0_1) \ge Ʊ^2{}_P(ß)$, $\forall\ ß \in Ř_2$, where $0_1$is identity element of $Ř_1$.

i. $Ʊ^2{}_O(\alpha - ß) = \max\{Ʊ^2{}_M(0_1 - 0_1), Ʊ^2{}_O(\alpha - ß)\}$
$= Ʊ^2{}_{MUO}(0_1 - 0_1, \alpha - ß)$
$= Ʊ^2{}_{MUO}((0_1, \alpha) - (0_1, ß))$
$\le \max\{Ʊ^2{}_{MUO}(0_1 - \alpha), Ʊ^2{}_{MUO}(0_1 - ß)\}$
$= \max\{\max\{Ʊ^2{}_M(0_1), Ʊ^2{}_O(\alpha)\}, \max\{Ʊ^2{}_M(0_1), Ʊ^2{}_O(ß)\}\}$
$= \max\{Ʊ^2{}_O(\alpha), Ʊ^2{}_O(ß)\}$
$Ʊ^2{}_O(\alpha - ß) \le \max\{Ʊ^2{}_O(\alpha), Ʊ^2{}_O(ß)\}$

ii. $Ʊ^2{}_O(\alpha ß) = \max\{Ʊ^2{}_M(0_1 0_1), Ʊ^2{}_O(\alpha ß)\}$
$= Ʊ^2{}_{MUO}(0_1 0_1, \alpha ß)$
$= Ʊ^2{}_{MUO}((0_1, \alpha).(0_1, ß))$
$\le \max\{Ʊ^2{}_{MUO}((0_1, \alpha), Ʊ^2{}_{MUO}(0_1, ß)\}$
$= \max\{\max\{Ʊ^2{}_M(0_1), Ʊ^2{}_O(\alpha)\}, \max\{Ʊ^2{}_M(0_1), Ʊ^2{}_O(ß)\}\}$
$= \max\{Ʊ^2{}_O(\alpha), Ʊ^2{}_O(ß)\}$
$Ʊ^2{}_O(\alpha ß) \le \max\{Ʊ^2{}_O(\alpha), Ʊ^2{}_O(ß)\}$

iii. $Ʊ^2{}_P(\alpha - ß) = \min\{Ʊ^2{}_N(0_1 - 0_1), Ʊ^2{}_P(\alpha - ß)\}$
$= Ʊ^2{}_{N\cap P}(0_1 - 0_1, \alpha - ß)$
$= Ʊ^2{}_{N\cap P}((\alpha, 0_1) - (ß, 0_1))$
$\ge \min\{(Ʊ^2{}_{N\cap P}(0_1, \alpha), Ʊ^2{}_{N\cap P}(0_1, ß)\}$
$= \min\{\min\{(Ʊ^2{}_N(0_1), Ʊ^2{}_P(\alpha)\}, \min\{Ʊ^2{}_N(0_1), Ʊ^2{}_P(ß)\}\}$
$= \{Ʊ^2{}_P(\alpha), Ʊ^2{}_P(ß)\}$
$Ʊ^2{}_P(\alpha - ß) \ge \min\{Ʊ^2{}_P(\alpha), Ʊ^2{}_P(ß)\}$

iv. $Ʊ^2{}_P(\alpha ß) = \min\{Ʊ^2{}_N(0_1 0_1), Ʊ^2{}_P(\alpha ß)\}$
$= Ʊ^2{}_{N\cap P}(0_1 0_1, \alpha ß)$
$= Ʊ^2{}_{N\cap P}((\alpha, 0_2)(ß, 0_2))$
$\ge \min\{Ʊ^2{}_{N\cap P}((0_1, \alpha), Ʊ^2{}_{N\cap P}((0_1, ß)\}$
$= \min\{\min\{Ʊ^2{}_N(0_1), Ʊ^2{}_P(\alpha)\}, \min\{Ʊ^2{}_N(0_1), Ʊ^2{}_P(\alpha)\}\}$
$= \min\{Ʊ^2{}_P(\alpha), Ʊ^2{}_P(ß)\}$
$Ʊ^2{}_P(\alpha ß) \ge \min\{Ʊ^2{}_P(\alpha), Ʊ^2{}_P(ß)\}$

Hence, Ȟ is an ĬMĂƑ subring of $Ř_2$

Remark 3.1.9

Suppose $Ɠ = \{(\alpha, Ʊ_M(\alpha), Ʊ_N(\alpha))/\alpha \in Ř1\}$ and $Ȟ = \{(ß, Ʊ_O(ß), Ʊ_P(ß))/ß \in Ř_2\}$ are ĬMĂƑS on a ring $Ř_1$and $Ř_2$ correspondingly. If the anti-Cartesian product $Ɠ \times Ȟ$ forms an ĬMĂƑ subring of a ring $Ř_1 \times Ř_2$, then it must be true that Ɠ is an ĬMĂƑ subring of a ring $Ř_1$or Ȟ is an ĬMĂƑ subring of a ring $Ř_2$.

Conclusion:

In the framework of Pythagorean multi-anti fuzzy rings, a significant and innovative extension of fuzzy algebraic structures, we present and rigorously describe the idea of the Cartesian product in this study. As a basic operation, the Cartesian product offers a mathematical foundation for systematically combining many intuitionistic multi-anti fuzzy sets, enabling the formation of increasingly intricate and composite fuzzy structures. The formal definition of this Cartesian product, together with a thorough examination of its theoretical characteristics and ramifications, are the main topics of this

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work. This study establishes the basis for new techniques for building and analyzing fuzzy algebraic systems that can manage higher degrees of complexity and ambiguity by applying the Cartesian product to such complicated fuzzy systems. The features of the Cartesian product in intuitionistic multi-anti fuzzy rings are studied in considerable detail in this study.

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