Dialogue Social Science Review (DSSR)
www.thedssr.com
ISSN Online: 3007-3154
ISSN Print: 3007-3146
Vol. 3 No. 11 (November) (2025)

Inequalities Of Hermite-Hadamard Like For $S-$Convex Function In The 2nd Kind And Implementations In Numerical Integration And Probability Theory

Faraz Mehmood
Department of Mathematics, Dawood University of Engineering and Technology, New M. A. Jinnah Road, Karachi, Pakistan and
Department of Mathematics, Samarkand State University, University boulevard 15, Samarkand 140104, Uzbekistan. Email: faraz.mehmood@duet.edu.pk

Muhammad Awais Shaikh
Department of Mathematics, University of Karachi, University Road, Karachi, Pakistan
Email:m.awaisshanikh2014@gmail.com

Syeda Ismat Zehra
Department of Mathematics, Dawood University of Engineering and Technology, New M. A. Jinnah Road, Karachi, Pakistan. Email: ismat.zehra@duet.edu.pk

Fouzia Abid
Department of Mathematics, Dawood University of Engineering and Technology, New M. A. Jinnah Road, Karachi, Pakistan. Email: fouzia.abid@duet.edu.pk

Mateen Ahmed Ghaznavi
IIEE (Institute of Industrial Electronics Engineering), University Road, Karachi, Pakistan. Email: mateenghaznavi@gmail.com

ABSTRACT
In the present article, we prove the Hermite-Hadamard (H-H) like inequalities for s−convex (conv.) in the 2nd kind and implementations for theory of Probability (Prob.) & numerical integration are deduced. Some consequences of several published articles would be captured as especial cases. Moreover, we deduce few especial cases of s−conv. function (func.).

Keywords: Convex func., H-H inequality (inequal.), Power-mean inequal., Hölder inequal., Numerical integration, Prob. density func.
2010 MSC. 26A51, 33B15, 26D10, 26D15, 26D20.

1. INTRODUCTION
About the features of convex functions, we code some lines from [24] 'Numerous problems in applied and pure mathematics involve convex functions. They act extremely crucial role in research of problems of non-linear and linear programming. The convex

Dialogue Social Science Review (DSSR)
www.thedssr.com
ISSN Online: 3007-3154
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Vol. 3 No. 11 (November) (2025)

functions' theory falls under the broader topic of convexity. However, this theory significantly affects practically every area of mathematical sciences. One of the earliest areas of mathematics where the concept of convexity is necessary for graphic analysis. Calculus provides us with a useful technique, the second derivative test, to identify convexity'.

We must generalize the idea of convex functions in order to generalize Ostrowski's inequality. In this way, we may quickly identify the generalizations and specific instances of the inequal. We recollect several definitions from the literature [2,6,7,8,9,11,12,15,16,17,18,19,20,21,22] for variety of convex functions.

Definition 1.1. A func. $g : K \subseteq (-\infty, \infty) \to (-\infty, \infty)$ is known as conv., if

$$g(\vartheta v + (1-\vartheta) w) \leq \vartheta g(v) + (1-\vartheta) g(w), \tag{1.1}$$

$\forall\, w, v \in K$, $\vartheta \in [0,1]$.

We remind term of $P-$conv. func. (see [3]).

Definition 1.2. A func. $g : K \subseteq (-\infty, \infty) \to (-\infty, \infty)$ is known as $P-$conv., if

$$g(\vartheta v + (1-\vartheta) w) \leq g(v) + g(w), \qquad g \geq 0$$

$\forall\, w, v \in K$, $\vartheta \in (0,1]$.

We remind term of quasi-conv. func. from [5].

Definition 1.3. A func. $g : K \subseteq (-\infty, \infty) \to (-\infty, \infty)$ is known as quasi-conv., if

$$g((1-\vartheta) w + \vartheta v) \leq \max\{g(w), g(v)\} \tag{1.2}$$

$\forall\, w, v \in K$, $\vartheta \in [0,1]$.

We remind terms $s-$conv. func. in the first & second kind (see to [23]).

Definition 1.4. A func. $g : K \subseteq [O,\infty) \to [O,\infty)$ is known as $s-$conv. in the 1st kind here $s \in (O,1]$, if

$$g(\vartheta v + (1-\vartheta) w) \leq \vartheta^{s} g(v) + (1-\vartheta^{s}) g(w),$$

$\forall\, w, v \in K$, $\vartheta \in [O,1]$.

Remark 1.4. If we include s = 0 in the above inequality then we obtain the refinement of quasi-conv. func.

Definition 1.5. A func. $g : K \subseteq [O,\infty) \to [O,\infty)$ is known as $s-$conv. in the 2nd kind here $s \in (O,1]$, if

$$g(\vartheta v + (1-\vartheta) w) \leq \vartheta^{s} g(v) + (1-\vartheta)^{s} g(w), \tag{1.3}$$

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Vol. 3 No. 11 (November) (2025)

$\forall\ w, v \in K, \vartheta \in [0,1]$.

Remark 1.4. If we include s = 0 in above inequal. then we obtain the $P-$conv. func.

In practically all scientific fields, inequalities play a major impact. Our primary target is on H–H like inequalities, despite the discipline's enormous scope.

The convexity theory is closely connected to the inequalities theory. Many well-known Inequalities in literature consequence directly from applying conv. functions. The H–H equation is a notable equation for conv. funct. that has been thoroughly researched in recent decades. It is discovered independently by Hadamard & Hermite and it is stated as follows: Any func. $g : K \subseteq (-\infty,\infty) \to (-\infty,\infty)$ be conv., $m, \ell \in K$ with $m > \ell$, If & only if

$$g\left(\frac{\ell+m}{2}\right) \leq \frac{1}{m-\ell}\int_{\ell}^{m} g(v)dv \leq \frac{g(\ell)+g(m)}{2} \qquad (1.4)$$

this is known as H-H inequal. Equation (1.4) has become a crucial pillar in the area of prob. & optimization. Additionally, numerous researchers have refined or generalized equation (1.4) for conv., s- conv., quasi- conv., & various other varieties of functions.

In [4], the following consequence had derived by Agarwal & Dragomir, which includes the H-H like integral inequal.

Proposition 1.5. *Let $g : K^o \subseteq (-\infty,\infty) \to (-\infty,\infty)$ be differentiable mapping (diff. map.) in the Interior K^o of K, here $m, \ell \in K^o$ along $m > \ell$. If $|g'|$ is conv. in interval $[\ell, m]$. Then the below equation holds*

$$\left|\frac{g(\ell)+g(m)}{2} - \frac{1}{m-\ell}\int_{\ell}^{m} g(u)du\right| \leq \frac{(m-\ell)(|g'(\ell)|+|g'(m)|)}{8} \qquad (1.5)$$

For additional recent consequences on H-H like inequalities involving various classes of conv. functions, refer to [1, 10, 13, 14, 25].

Kavurmaci et al. [10] established few novel inequalities of H-H like for conv. functions & implementations by utilizing Hölder inequality and Powermean inequal.

The primary goal of the article is to generalize few H-H like inequalities to $s-$conv. func. in the 2^{nd} kind by employing the Hölder & Powermean inequalities. The implementations also encompass areas such as prob. & numerical integration. We will capture some findings of various articles [4, 10] and also examine especial cases of $s-$conv. func.

2. GENERALIZATION OF HERMITE-HADAMARD LIKE INEQUALITIES

Regarding proof of our primary findings, below Lemma (see [10]) is required.

Lemma 2.1. *Suppose* $g: K \subseteq (-\infty, \infty) \rightarrow (-\infty, \infty)$ *is diff. map. in interval* $K^0 \subset (-\infty, \infty)$ *here* $m, \ell \in K$ *along* $m > \ell$, *if* $g' \in L[\ell, m]$, *then*

$$\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{\ell}^{m} g(u)du$$

$$= \frac{(v-\ell)^2}{m-\ell}\int_0^1 (1-\vartheta)g'(\vartheta v + (1-\vartheta)\ell)d\vartheta + \frac{(m-v)^2}{m-\ell}\int_0^1 (1-\vartheta)g'(\vartheta v + (1-\vartheta)m)d\vartheta.$$

The below consequences may be derived by employing lemma 2.1

Theorem 2.2. *Suppose* $g: K \subseteq (-\infty, \infty) \rightarrow (-\infty, \infty)$ *is diff. map. in interval* $K^0 \subset (-\infty, \infty)$ ε $g' \epsilon L[\ell, m]$, *here* $m, \ell \in K$ *along* $m > \ell$, *if* $|g'|$ *is s−conv. in the* 2^{nd} *kind on* $[\ell, m]$, *then*

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{\ell}^{m} g(u)du\right|$$

$$\leq \frac{(v-\ell)^2}{m-\ell}\left[\frac{|g'(v)|}{(s+1)(s+2)} + \frac{|g'(\ell)|}{(s+2)}\right] + \frac{(m-v)^2}{m-\ell}\left[\frac{|g'(v)|}{(s+1)(s+2)} + \frac{|g'(m)|}{(s+2)}\right].$$

For every $g \epsilon [\ell, m]$.

Proof. Utilizing lemma 2.1 & taking Modulus, then

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{\ell}^{m} g(u)du\right|$$

$$\leq \frac{(v-\ell)^2}{m-\ell}\int_0^1 (1-\vartheta)|g'((1-\vartheta)\ell + \vartheta v)|d\vartheta + \frac{(m-v)^2}{m-\ell}\int_0^1 (1-\vartheta)|g'(\vartheta v + (1-\vartheta)m)|d\vartheta$$

$\because$ $|g'|$ is s−conv. in the 2^{nd} kind, then

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell} - \frac{1}{m-\ell}\int_{\ell}^{m} g(u)du\right|$$

$$\leq \frac{(v-\ell)^2}{m-\ell}\int_0^1 (1-\vartheta)[\vartheta^s|g'(v)| + (1-\vartheta)^s|g'(\ell)|]\,d\vartheta$$

$$+ \frac{(m-v)^2}{m-\ell}\int_0^1 (1-\vartheta)\,[\vartheta^s|g'(v)| + (1-\vartheta)^s|g'(m)|]d\vartheta$$

$$= \frac{(v-\ell)^2}{m-\ell}\left[\frac{|g'(v)|}{(s+1)(s+2)} + \frac{|g'(\ell)|}{(s+2)}\right] + \frac{(m-v)^2}{m-\ell}\left[\frac{|g'(v)|}{(s+1)(s+2)} + \frac{|g'(m)|}{(s+2)}\right]$$

thus completing the proof.

Remark 2.3. The *P*−conv. func. is obtained If choose $s = 0$ in Theorem 2.2.

Remark 2.4. We attain the Theorem 4 of [10] If choose $s = 1$ in Theorem 2.2.

Corollary 2.5. *In Theorem* 2.2, *choosing* $v = \frac{\ell+m}{2}$ *we get*

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$$\left|\frac{g(\ell)+g(m)}{2}-\frac{1}{m-\ell}\int_{\ell}^{m}g(u)du\right|\leq\frac{m-\ell}{4}\left[\frac{2\left|g'\left(\frac{\ell+m}{2}\right)\right|}{(s+1)(s+2)}+\left(\frac{|g'(m)|+|g'(\ell)|}{(s+2)}\right)\right]$$

Remark 2.6. Few Remarks regarding Corolary 2.5 are below as especial cases.

(i) By utilizing the convexity property of $|g'|$ in Corollary 2.5, we get established equation (1.5) (capture theorem 2.2 of article [4]).

(ii) The $P-$conv. func. is obtained If choose $s = 0$ in Corollary 2.5.

Remark 2.7. We capture the Corollary 2 of article [10] If choose $s = 1$ in Corollary 2.5.

Theorem 2.8. *Suppose* $g : K \subseteq (-\infty,\infty) \to (-\infty,\infty)$ *is diff. map. in* $K^{o} \subset (-\infty,\infty)$ *ε* $g' \epsilon L[\ell,m]$, *here* $\ell, m \in K$ *along* $\ell < m$. *If* $|g'|^{\frac{p}{p-1}}$ *is s−conv. in 2nd kind in the interval* $[\ell, m]$ *& for some fixed* $1 < q$. *Then*

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell}-\frac{1}{m-\ell}\int_{\ell}^{m}g(u)du\right|$$

$$\leq\frac{1}{m-\ell}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[(v-\ell)^{2}(|g'(v)|^{q}+|g'(\ell)|^{q})^{\frac{1}{q}}\right.$$
$$\left.+(m-v)^{2}(|g'(v)|^{q}+|g'(m)|^{q})^{\frac{1}{q}}\right]$$

for every $g \in [\ell, m]$.

Proof. Utilizing Lemma 2.1 & $s-$convexity in the 2nd kind of $|g'|$ & then implementing the widely recognized Hölder inequality, we get

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell}-\frac{1}{m-\ell}\int_{\ell}^{m}g(u)du\right|$$
$$\leq\frac{(v-\ell)^{2}}{m-\ell}\int_{0}^{1}(1-\vartheta)|g'(\vartheta v+(1-\vartheta)\ell|d\vartheta$$
$$+\frac{(m-v)^{2}}{m-\ell}\int_{0}^{1}(1-\vartheta)|g'(\vartheta v+(1-\vartheta)m)|d\vartheta$$

$$\leq\frac{(v-\ell)^{2}}{m-\ell}\int_{0}^{1}(1-\vartheta)\left[\vartheta^{s}|g'(v)|+(1-\vartheta)^{s}|g'(\ell)|\right]d\vartheta+\frac{(m-v)^{2}}{m-\ell}\int_{0}^{1}(1-\vartheta)\left[\vartheta^{s}|g'(v)|+(1-\vartheta)^{s})|g'(m)|\right]d\vartheta$$

$$\leq\frac{(v-\ell)^{2}}{m-\ell}\left(\int_{0}^{1}(1-\vartheta)^{p}d\vartheta\right)^{\frac{1}{p}}\left[\int_{0}^{1}\left(\vartheta^{s}|g'(v)|+(1-\vartheta)^{s}|g'(\ell)|\right)^{q}d\vartheta\right]^{\frac{1}{p}}$$

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www.thedssr.com
ISSN Online: 3007-3154
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$$+\frac{(m-v)^2}{m-\ell}\left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}}\left[\int_0^1 \left(\vartheta^s|g'(v)|+(1-\vartheta)^s|g'(m)|\right)^q d\vartheta\right]^{\frac{1}{p}}$$

$$\leq \frac{(v-\ell)^2}{m-\ell}\left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}}\left[\int_0^1 \vartheta^s|g'(v)|^q d\vartheta+\int_0^1 (1-\vartheta)^s|g'(\ell)|^q d\vartheta\right]^{\frac{1}{p}}$$

$$+\frac{(m-v)^2}{m-\ell}\left(\int_0^1 (1-\vartheta)^p d\vartheta\right)^{\frac{1}{p}}\left[\int_0^1 \vartheta^s|g'(v)|^q d\vartheta+\int_0^1 (1-\vartheta)^s|g'(m)|^q d\vartheta\right]^{\frac{1}{p}}$$

$$\leq \frac{1}{m-\ell}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\Big[(v-\ell)^2(|g'(v)|^q+|g'(\ell)|^q)^{\frac{1}{q}}$$
$$+(m-v)^2(|g'(v)|^q+|g'(m)|^q)^{\frac{1}{q}}\Big]$$

Remark 2.9. The P−conv. func. is obtained If choose s =0 in Theorem 2.8.

Remark 2.10. We attain the Theorem 5 of [10] If choose s = 1 in Theorem 2.8.

Corollary 2.11. *In Theorem* 2.8, *choosing* $v=\frac{\ell+m}{2}$ *we get*

$$\left|\frac{g(\ell)+g(m)}{2}-\frac{1}{m-\ell}\int_\ell^m g(u)du\right|\leq\frac{m-\ell}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left(|g'(\ell)|^q+\left|g'\left(\frac{\ell+m}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(|g'(m)|^q+\left|g'\left(\frac{\ell+m}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

Remark 2.12. The P−Convex function is obtained If choose s = 0 in Corollary 2.11.

Remark 2.13. We capture the Corollary 3 of article [10] If choose s = 1 in Corollary 2.11.

Theorem 2.14. *Suppose* $g: K\subseteq(-\infty,\infty)\to(-\infty,\infty)$ *is diff. map. on* $K^o\subset(-\infty,\infty)$ *ε* $g'\epsilon L[\ell,m]$, *here* $\ell,m\in K$ *along* $\ell<m$. *If* $|g'|^q$ *is s−conv. in* 2nd *kind on* $[\ell,m]$ *& for some fixed* $q\geq 1$. *Then*

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell}-\frac{1}{m-\ell}\int_\ell^m g(u)du\right|$$

$$\leq\frac{1}{2^{1-\frac{1}{q}}(m-\ell)}\left[(v-\ell)^2\left(\frac{|g'(v)|^q}{(s+1)(s+2)}+\frac{|g'(\ell)|^q}{(s+2)}\right)^{\frac{1}{q}}\right.$$

$$+(m-v)^2\left(\frac{|g'(v)|^q}{(s+1)(s+2)}+\frac{|g'(m)|^q}{(s+2)}\right)^{\frac{1}{q}}\Bigg]$$

for every $g\epsilon\,[\ell,m], q=\frac{p}{p-1}$

Proof. Consider $1\leq q$ & Utilizing Lemma 2.1 & implementing the widely recognized power-mean inequality, we get

$$\left|\frac{(m-v)g(m)+(v-\ell)g(\ell)}{m-\ell}-\frac{1}{m-\ell}\int_\ell^m g(u)du\right|$$

$$\leq\frac{(v-\ell)^2}{m-\ell}\int_0^1(1-\vartheta)|g'(\vartheta v+(1-\vartheta)\ell)|d\vartheta$$

$$+\frac{(m-v)^2}{m-\ell}\int_0^1(1-\vartheta)|g'(\vartheta v+(1-\vartheta)m|d\vartheta$$

$$\leq\frac{(v-\ell)^2}{m-\ell}\left(\int_0^1(1-\vartheta)d\vartheta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\vartheta)|g'(\vartheta v+(1-\vartheta)\ell|^q d\vartheta\right)^{\frac{1}{q}}$$

$$+\frac{(m-v)^2}{m-\ell}\left(\int_0^1(1-\vartheta)d\vartheta\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\vartheta)|g'(\vartheta v+(1-\vartheta)m|^q d\vartheta\right)^{\frac{1}{q}}$$

∵ $|g'|$ is s–conv. in the 2nd kind & first we taking term

$$\int_0^1(1-\vartheta)|g'(\vartheta v+(1-\vartheta)\ell|^q d\vartheta$$

$$\leq\int_0^1(1-\vartheta)\left[\vartheta^s|g'(v)|^q+(1-\vartheta)^s|g'(\ell)|^q\right]d\vartheta$$

$$=\frac{|g'(v)|^q}{(s+1)(s+2)}+\frac{|g'(\ell)|^q}{(s+2)}$$

Analogously,

$$\int_0^1(1-\vartheta)|g'((1-\vartheta)m+\vartheta v)|^q d\vartheta\leq\frac{|g'(v)|^q}{(s+1)(s+2)}+\frac{|g'(m)|^q}{(s+2)}$$

We obtain the desired result by combining all above inequalities.

Remark 2.15. The P–conv. func. is obtained If choose s =0 in Theorem 2.14.
Remark 2.16. We attain the Theorem 7 of article [10] If choose s = 1 in Theorem 2.14.

Corollary 2.17. *In* Theorem 2.14, *choosing* $v=\frac{\ell+m}{2}$ *we get*

$$\left|\frac{g(\ell)+g(m)}{2}-\frac{1}{m-\ell}\int_{\ell}^{m}g(u)du\right|$$
$$\leq\frac{(m-\ell)2^{\frac{1}{q}}}{8}\left[\left(\frac{\left|g'\left(\frac{\ell+m}{2}\right)\right|^{q}}{(s+2)(s+1)}+\frac{|g'(\ell)|^{q}}{(s+2)}\right)^{1/q}+\left(\frac{\left|g'\left(\frac{\ell+m}{2}\right)\right|^{q}}{(s+1)(s+2)}+\frac{|g'(k)|^{q}}{(s+2)}\right)^{1/q}\right]$$

Remark 2.18. The P−conv. func. is obtained If choose s =0 in Corollary 2.17.

Remark 2.19. We capture the Corollary 4 of article [10] If choose s = 1 in Corollary 2.17.

3. IMPLEMENTATION TO NUMERICAL INTEGRATION

3.1. **The Trapezoidal Formula.** Suppose $d : \ell = \theta_0 < \theta_1 < \cdots < \theta_n = m$ is division of interval $[\ell, m]$ & $h_i = \theta_{i+1} - \theta_i$, $(i = 0,1,2,\cdots ,n - 1)$ & consider the quadrature formula

$$\int_{\ell}^{m} g(u)du = Q(g,d) + R(g,d), \qquad (3.1)$$

where

$$Q(g,d)=\sum_{i=0}^{n-1}((\theta_{i+1}-v)g(\theta_{i+1})+(v-\theta_i)g(\theta_i))$$

for the trapezoidal version & $R(g,d)$ represents the associated approximation error.

Theorem 3.1. *With all the suppositions of Theorem* 2.2, *for every division* d *of the interval* $[\ell, m]$. *Then in equation (3.1), the trapezoidal error estimate satisfies:*

$$|R(g,d)|$$
$$\leq\sum_{i=0}^{n-1}(v-\theta_i)^2\left[\frac{|g'(v)|}{(s+1)(s+2)}+\frac{|g'(\theta_i)|}{(s+2)}\right]+\sum_{i=0}^{n-1}(\theta_{i+1}-v)^2\left[\frac{|g'(v)|}{(s+1)(s+2)}+\frac{|g'(\theta_{i+1})|}{(s+2)}\right]$$

for every $g \in [\ell, m]$.

Proof. Utilizing theorem 2.2 on sub-interval $[\theta_i, \theta_{i+1}]$ here $i = 0,1,2 ,..., n - 1$ of the division, we get

$$\left|\frac{(\theta_{i+1}-v)g(\theta_{i+1})+(v-\theta_i)g(\theta_i)}{h_i}-\frac{1}{h_i}\int_{\theta_i}^{\theta_{i+1}}g(u)du\right|$$
$$\leq\frac{(v-\theta_i)^2}{h_i}\left[\frac{|g'(v)|}{(s+1)(s+2)}+\frac{|g'(\theta_i)|}{(s+2)}\right]+\frac{(\theta_{i+1}-v)^2}{h_i}\left[\frac{|g'(v)|}{(s+1)(s+2)}+\frac{|g'(\theta_{i+1})|}{(s+2)}\right]$$

Now we take summation over i in equation (3.1) from 0 to $n - 1$, then

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www.thedssr.com
ISSN Online: 3007-3154
ISSN Print: 3007-3146
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$$\left|\int_j^k g(u)du - Q(g,d)\right| = \left|\sum_{i=0}^{n-1}\left[\int_{\theta_i}^{\theta_{i+1}} g(u)du - ((\theta_{i+1}-\mathcal{v})g(\theta_{i+1}) - (\mathcal{v}-\theta_i)g(\theta_i))\right]\right|$$

$$\leq \sum_{i=0}^{n-1}\left|\int_{\theta_i}^{\theta_{i+1}} g(u)du - \left((\theta_{i+1}-\mathcal{v})g(\theta_{i+1}) - (\mathcal{v}-\theta_i)g(\theta_i)\right)\right|$$

$$\leq \sum_{i=0}^{n-1}(\mathcal{v}-\theta_i)^2\left[\frac{|g\prime(\mathcal{v})|}{(s+1)(s+2)} + \frac{|g'(\theta_i)|}{(s+2)}\right] + \sum_{i=0}^{n-1}(\theta_{i+1}-\mathcal{v})^2\left[\frac{|g\prime(\mathcal{v})|}{(s+1)(s+2)} + \frac{|g'(\theta_{i+1})|}{(s+2)}\right]$$

which completes the proof.

Remark 3.2. We can also derive above similar findings for Theorem 2.14, Corollary 2.17, Corollary 2.11, Theorem 2.8 & Corollary 2.5.

Remark 3.3. Entire Remarks of 2nd section are also hold for Remark 3.2 & Theorem 3.1.

4. IMPLEMENTATION TO PROBABILITY THEORY

Suppose $\mathfrak{R}$ is a random variable taking values in the finite $[\ell, m]$ with the prob. density func. $g : [\ell, m] \rightarrow [0,1]$ & the cumulative distribution func. $\mathcal{G}(\mathcal{v}) = P(\mathfrak{R} \leq \mathcal{v}) = \int_\ell^m g(u)du$.

Theorem 4.1. *With all the assumptions of Theorem* 2.2, *then*

$$\left|\frac{(m-\mathcal{v})\mathcal{G}(m) + (\mathcal{v}-\ell)\mathcal{G}(\ell)}{m-\ell} - \frac{m-\mathcal{E}(\mathfrak{R})}{m-\ell}\right|$$

$$\leq \frac{(\mathcal{v}-\ell)^2}{m-\ell}\left[\frac{|\mathcal{G}\prime(\mathcal{v})|}{(s+1)(s+2)} + \frac{|\mathcal{G}\prime(\ell)|}{(s+2)}\right] + \frac{(m-\mathcal{v})^2}{m-\ell}\left[\frac{|\mathcal{G}\prime(\mathcal{v})|}{(s+1)(s+2)} + \frac{|\mathcal{G}\prime(m)|}{(s+2)}\right] \qquad (4.1)$$

for every $g \in [\ell, m]$. Here $\mathcal{E}(\mathfrak{R})$ is the expectation of $\mathfrak{R}$.

Proof. Choose $g = \mathcal{G}$, we attain (4.1), by implementing the below identity in Theorem 2.2.

$$\mathcal{E}(\mathfrak{R}) = \int_\ell^m u\mathcal{G}(u)du = m - \int_\ell^m \mathcal{G}(u)du.$$

$\because \mathcal{G}(\ell) = 0$ & $\mathcal{G}(m) = 1$.

Remark 4.3. Entire Remarks of 2nd section are also hold for Remark 4.2 & Theorem 4.1.

5. CONCLUSION

In the current article, the famous H-H like inequalities for $s-$conv. func. in the 2nd kind are generalized by implementing the widely recognized power-mean & Hölder's inequalities & implementations for theory of prob. & numerical integration are also deduced. We have captured various findings of several published articles [4,10] & also deduced few especial cases of $s-$conv. func. in the 2nd kind.

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